Acceleration of particles as a universal property of ergosphere

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We show that recent observation made in Grib and Pavlov, arXiv:1301.0698 for the Kerr black hole is valid in the general case of rotating axially symmetric metric. Namely, collision of two particles in the ergosphere leads to indefinite growth of the energy in the centre of mass frame, provided the angular momentum of one of two particles is negative and increases without limit for a fixed energy at infinity. General approach enabled us to elucidate, why the role of the ergosphere in this process is crucial.

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I. INTRODUCTION

In recent years, the interest to high-energetic processes near black holes revived due to observation [1] made by Bañados, Silk, and West (hereafter, BSW effect), so several dozens of papers appeared since 2009. These authors found that under certain conditions, the energy $E_{c.m.}$ in the centre of mass frame of two particles colliding near a black hole can grow unbound. Meanwhile, potential attempts to observe the consequences of the BSW effect are faced with difficulties since the strong red shift restricts the energies of possible outcome which can be detected at infinity [2] - [4]. Quite recently, a new mechanism of getting ultra-high energies in black hole physics was suggested in [5]. Grib and Pavlov demonstrated there that inside the ergosphere of the Kerr black hole, $E_{c.m.}$ also grows indefinitely if the angular momentum of one of particles takes large negative values. Then, the aforementioned problems with detecting at infinity do not arise since collision occurs, in general, outside

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the horizon. For such an effect, there is no requirement to fine-tune the parameters of one particle in contrast to the BSW one [1] that also facilitates the realization of the phenomenon under discussion. It is worth stressing that interest to the potential role of the ergosphere in the high energy processes revived recently in connection with the problem of jet collimation [8], [9].

The main goal of the present work is to show that the effect of indefinite growth of $E_{c.m.}$ occurs inside the ergosphere of a generic "dirty" (surrounded by matter) stationary rotating axially symmetric black hole. We also elucidate the role of the ergoregion in the effect under discussion. This is done in a general form, without resorting to the properties of a particular metric. In the original paper [1], the BSW effect was considered for extremal black holes and it took some efforts to show that a similar effect can occur also near nonextremal ones [6]. For the process in question [5], it is clear from the very beginning that a black hole can be extremal or nonextremal. Moreover, as is clear from consideration below, the horizon can be absent at all but the existence of the ergosphere is mandatory (examples of spacetimes with the ergosphere but without the horizon are discussed in [7]).

We use the systems of units in which the fundamental constants $G = c = 1$.

II. FORM OF METRIC AND GEODESICS EQUATIONS

Let us consider the metric

$$ds^2 = -N^2dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + g_{\theta\theta}d\theta^2. \tag{1}$$

Here, the metric coefficients do not depend on t and ϕ. On the horizon $N = 0$. In (1), the factor $\Delta(r) \sim N^2$ is singled out for convenience. The coefficient ρ can depend on θ. The Kerr metric belongs just to this class. We assume that, similarly to the Kerr metric,

$$\omega > 0 \tag{2}$$

everywhere.

If a neutral particle of the mass m moves freely in the spacetime (1), its energy $E = -mu_0$ and the angular momentum $L = mu_\phi$ are conserved due to the existence of Killing vectors responsible for time translation and rotation. Here, $u^\mu = \frac{df^\mu}{d\tau}$ is the four-velocity, $x^\mu = (t, \phi, r, \theta)$. Then, the equators of motion along geodesics read

$$m\dot{t} = mu^0 = \frac{X}{N^2},\ X = E - \omega L. \tag{3}$$

We assume the forward in time condition $\dot{t} > 0$, so that

$$E - \omega L \geq 0 \tag{4}$$

should be satisfied. Then, it follows from the geodesic equations and the normalization condition $u_\mu u^\mu = -1$ that

$$m\dot{\phi} = \frac{L}{g} + \frac{\omega X}{N^2},\ g = g_{\phi\phi} \tag{5}$$

that can be also rewritten as

$$m\dot{\phi} = \frac{\omega E}{N^2} - L\frac{g_{00}}{gN^2}, \tag{6}$$

$$\frac{\rho}{\sqrt{\Delta}} m\dot{r} = \varepsilon \frac{Z}{N}, \tag{7}$$

where dot denotes the derivative with respect to τ. Here,

$$Z^2 = X^2 - N^2(m^2 + \frac{L^2}{g} + m^2 g_{\theta\theta}\dot{\theta}^2), \tag{8}$$

$\varepsilon = sign\dot{r}$.

Using the relation

$$g_{00} = -N^2 + \omega^2 g, \tag{9}$$

we can rewrite eq. (8) as

$$Z^2 = \frac{L^2 g_{00}}{g} - 2E\omega L + E^2 - N^2 m^2 (1 + B),\ B = g_{\theta\theta}\dot{\theta}^2 \geq 0, \tag{10}$$

or

$$Z^2 = \frac{g_{00}}{g}(L_- - L)(L_+ - L), \tag{11}$$

where L_+ and L_- are the roots of the equation $Z = 0$. According to (7), this corresponds to turning points $\dot{r} = 0$. For the Kerr metric, the variables in the equations for geodesics are separated, and the values L_+ and L_- can be expressed in terms of the Carter constant - see eqs. (10) and (18) of [5]. For generic dirty black holes, the roots L_+ and L_- depend on $\dot{\theta}$ and cannot be found in the closed form in general. This can be done for equatorial motion, $\dot{\theta} = 0 = B$. Then, we find from (11)

$$L_\pm = \frac{g\omega}{g_{00}}E \pm \frac{N\sqrt{g}}{g_{00}}\sqrt{E^2 + m^2 g_{00}}. \tag{12}$$

III. ERGOSPHERE AND INTEGRALS OF MOTION

By definition, the surface of the ergosphere is defined by equation $g_{00} = 0$. We will discuss some general properties of three regions separately. For what follows, we also need the asymptotic expressions for Z for large $|L|$. It is clear from (2) and (4) that the limit $|L| \to \infty$ with fixed E can be realized for $L = -|L| < 0$ only.

A. Outside ergosphere, $g_{00} < 0$

Then, it follows from (10), the condition $Z^2 \geq 0$ and (4) that

$$E \geq \omega L + \sqrt{\omega^2 L^2 + N^2 m^2 (1+B) + \frac{L^2 |g_{00}|}{g}} \geq 0, \tag{13}$$

In terms of angular momenta, $L_+ \leq L \leq L_-$. It is seen from (10) that for $L \to -\infty$ and E fixed, the first term is negative and dominates Z^2, so this limit cannot be realized in this region.

B. Boundary of ergosphere, $g_{00} = 0$

Then, it follows from (6) that $m\dot{\phi} = \frac{\omega E}{N^2}$ does not depend on L for a fixed energy. In (11), the terms of the order L^2 in Z^2 cancel. For $|L_2| \to \infty$ it turns out that

$$Z \approx \sqrt{2E\omega |L|}, \tag{14}$$

where $E \geq 0$ according to (13).

C. Inside ergosphere, $g_{00} > 0$

If $L = -|L| < 0$, the second term in (6) is positive. Thus for a fixed energy the angular velocity of rotation is increasing function of $|L|$. The more negative becomes the angular momentum, the larger the angular velocity of rotation in the positive direction! This generalizes observation made in [5] for the Kerr metric. For large $|L|$,

$$Z = |L| \sqrt{\frac{g_{00}}{g}} + O(1). \tag{15}$$

Thus we see that trajectories with fixed E and $L=-\left|L\right|\rightarrow -\infty$ are possible inside the ergosphere or on its boundary but are forbidden outside it. This generalizes the corresponding property of the Kerr metric [5].

IV. ENERGY IN CENTRE OF MASS FRAME

For two colliding particles, in the point of collision one can define the energy in the centre of mass frame as

$$E_{c.m.}^{2}=-\left(m_{1}u_{1}^{\mu }+m_{2}u_{2}^{\mu }\right) \left(m_{1}u_{1\mu }+m_{2}u_{2\mu }\right) \tag{16}$$

that is the counterpart of the standard textbook formula for one particle $m^{2}=-p^{\mu }p_{\mu }$ where $p^{\mu }$ is the four-momentum.

Then,

$$E_{c.m.}^{2}=m_{1}^{2}+m_{2}^{2}+2m_{1}m_{2}\gamma \tag{17}$$

where the Lorentz factor of relative motion

$$\gamma =-u_{1}^{\mu }u_{2\mu }, \tag{18}$$

$\left(u^{\mu }\right) _{i}$ is the four-velocity of the i-th particle (i=1,2). Using (3) - (7) we obtain from (18) that

$$\gamma =\frac{1}{m_{1}m_{2}}(c-d)--g_{\theta \theta }\dot{\theta}_{1}\dot{\theta}_{2}\text{, }c=\frac{X_{1}X_{2}-\varepsilon _{1}\varepsilon _{2}Z_{1}Z_{2},}{N^{2}}\text{, }d=\frac{L_{1}L_{2}}{g}. \tag{19}$$

V. COLLISIONS WITH LARGE NEGATIVE ANGULAR MOMENTUM

We are interested in the situation when $E_{c.m.}^{2}$ can be made as large as possible. One possibility is to make denominator in c (19) small due to $N\rightarrow 0$. This corresponds to the BSW effect. Meanwhile, there is one more option to which Grib and Pavlov paid attention [5] - the case of large numerator. Let L_{1} be finite but the absolute value of L_{2} be very large (by analogy with the BSW effect, we call particle 1 usual and particle 2 critical). As is explained in Sec. III, this is only possible if (i) $L_{2}=-\left\vert L_{2}\right\vert <0$, (ii) collisions occur inside the ergosphere or near its boundary.

A. Collisions inside ergosphere

In the limit $|L_2| \to \infty$, using (11) and retaining in (19) the leading terms in L_2, we obtain

$$E_{c.m.}^2 = \frac{2\,|L_2|}{N^2}\alpha + O(1), \tag{20}$$

$$\alpha = \omega X_1 - \varepsilon_1 \varepsilon_2 Z_1 \sqrt{\frac{g_{00}}{g}} + \frac{L_1 N^2}{g} > 0 \tag{21}$$

Taking into account the useful relation

$$L_+ + L_- = \frac{2E}{g_{00}} g\omega \tag{22}$$

that follows from (10), one can check that $2\alpha = \frac{g_{00}}{g}[\varepsilon_1\sqrt{(L_1)_+ - L_1} - \varepsilon_2\sqrt{(L_1)_- - L_1}]^2$, whence

$$E_{c.m.}^2 \approx \frac{2\,|L_2|\,g_{00}}{N^2 g}[\varepsilon_1\sqrt{(L_1)_+ - L_1} - \varepsilon_2\sqrt{(L_1)_- - L_1}]^2. \tag{23}$$

It is seen from (23) that it is the property $g_{00} > 0$ inside the ergosphere that ensures the positivity of $E_{c.m.}^2$.

B. Collisions near the boundary of ergosphere

On the boundary of the ergosphere $g_{00} = 0$, the approximate expressions (20), (21) are not valid. Using (14), one obtains

$$E_{c.m.}^2 = \frac{2}{N^2}(\omega E_1\,|L_2| - Z_1 \varepsilon_1 \varepsilon_2 \sqrt{2E_2\omega\,|L_2|}) + O(1). \tag{24}$$

Thus inside the ergosphere and on the boundary the leading term has the order $|L_2|$ but the subleading terms are different: they are finite inside and proportional to $\sqrt{|L_2|}$ on the boundary.

VI. CASE OF CIRCULAR ORBITS

There is also a special case when $\theta = \frac{\pi}{2}$ and L_2 is adjusted to ensure the circular character of the orbit on which $\dot{r}_2 = 0 = Z_2$. Then, we have from (17) and (19) that

$$E_{c.m.}^2 = m_1^2 + m_2^2 + 2\left(\frac{X_1 X_2}{N^2} - \frac{L_1 L_2}{g}\right). \tag{25}$$

Let us assume that we are outside the ergosphere, so in our notations (which are here opposite to those in [5]) $L_2 = (L_2)_+ < (L_2)_-$. If we choose the radius of the orbit closer and closer to the boundary of the ergosphere, $g_{00} \to -0$. Then, it follows from (12) that

$$L_2 \approx -\frac{2NE_2\sqrt{g}}{|g_{00}|} \to -\infty, \tag{26}$$

whence

$$E^2_{c.m.} \approx -\frac{4E_1E_2\sqrt{g}\omega}{g_{00}N} \tag{27}$$

diverges as the boundary is approached. It is worth noting that the leading terms in (24) and (27) coincide but the subleading ones are different since Z_2 is large in the first case and vanishes in the second one.

Another case of interest arises when particle 2 is orbiting with $E_2 = -|E_2|$ from inside. Then, in a similar way, $L_2 = L_- \approx -\frac{2N|E_2|\sqrt{g}}{|g_{00}|}$. For particle 1, L_1 is finite and the energy obeys inequality (13) in which the term with g_{00} is neglected. Then, eq. (27) is valid in which, however, now $g_{00} \to +0$.

VII. COMPARISON WITH THE KERR METRIC

In this case, in the Boyer-Lindquiste coordinates,

$$\omega = \frac{2aMr}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}, \tag{28}$$

$$g = \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta], \tag{29}$$

$$N^2 = \frac{\rho^2\Delta}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}, \tag{30}$$

$$g_{00} = -(1 - \frac{2Mr}{\rho^2}), \tag{31}$$

$$\rho^2 = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2. \tag{32}$$

Here, M is the black hole mass, $a = \frac{J}{M}$ where J is the black hole angular momentum. Then, one can check that eq. (23) turns into eq. (25) of [5]. It is worth noting that the asymptotic form (23) is valid independently of special symmetry properties of the metric which were used in [5] for the Kerr case.

VIII. KINEMATIC EXPLANATION

The indefinitely growing energy in the centre of mass frame implies that the relative velocity tends to that of light. This occurs when one of particles has the velocity close to that of light whereas the other one moves with some finite velocity. For the BSW effect, such an explanation was given in [10]. It is based on the relationship (see eq. (15) of [10])

$$E - \omega L = \frac{mN}{\sqrt{1 - V^2}}, \tag{33}$$

where V is the local velocity of a particle in the zero angular momentum frame [11]. In the horizon limit $N \to 0$, the velocity $V_{us} \to 1$ if $E - \omega L > 0$ does not vanish on the horizon (a usual particle) and $V_{cr} < 1$ if $E - \omega L = 0$ on the horizon (the critical one).

Now, the situation is different since inside the ergosphere or on its boundary $N \neq 0$ in general (the pole points are exception but in their vicinity the ergosphere approaches the horizon, so standard BSW effect takes place). Instead, now for the critical particle the second term in the left hand side of (33) tends to infinity, so that the right hand side must also diverge, whence $V_{cr} \to 1$. For a usual particle, V takes some finite value $V_{us} < 1$. Thus instead of $V_{cr} < 1$, $V_{us} \to 1$ for the BSW effect, now, vice versa, $V_{cr} \to 1$ and $V_{us} < 1$ for collisions in or on the ergosphere.

The case under discussion possesses two interesting features: (i) $|L_2| \to \infty$ but E_2 is finite, (ii) this effect happens inside the ergosphere or on its boundary and is impossible outside it.

IX. CONCLUSION

Thus we suggested model-independent approach and generalized an interesting observation by Grib and Pavlov [5] to arbitrary "dirty" (surrounded by matter) rotating axially symmetric black holes. Some relevant properties of motion of geodesic particles are elucidated depending on their angular momentum and its location (outside, inside or on the boundary of the ergosphere). We also gave general explanation of the role of the ergosphere in the process of collisions with indefinite growth of $E_{c.m.}$ due to large negative angular momentum of the critical particle. As is clear from this approach, such an effect occurs even without black hole horizons, provided the ergosphere exists.

Thus the possibility of acceleration of particles to ultra-high energies turns out to be a universal property of the ergosphere.

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[1] Bañados M., Silk J. and West S.M., *Phys. Rev. Lett.* **103** (2009) 111102 [arXiv:0909.0169].

[2] Bejger M., Piran T., Abramowicz M., and Håkanson F., *Phys. Rev. Lett.* **109** (2012) 121101 [arXiv:1205.4350].

[3] Harada T., Nemoto H. and Miyamoto U., *Phys. Rev.* D **86** (2012) 024027 [Erratum ibid. D 86 (2012) 069902] [arXiv:1205.7088].

[4] Zaslavskii O. B., *Phys. Rev.* D **86** (2012) 084030 [arXiv:1205.4410].

[5] Grib A.A. and Pavlov Yu.V., arXiv:1301.0698.

[6] Grib A.A. and Pavlov Yu.V., *JETP Letters*, 92 (2010) 125.

[7] Glass E.N. and Krisch J.P., *Class. Quant. Grav.*, 21 (2004) 5543 [arXiv:gr-qc/0410089].

[8] J. Gariel, M. A. H. MacCallum, G. Marcilhacy, and N. O. Santos, *Astronomy and Astrophysics*, **515** (2010) A15 [arXiv:gr-qc/0702123].

[9] J. A. de Freitas Pacheco, J. Gariel, G. Marcilhacy, and N. O. Santos, The Astrophysical Journal **759** (2012) 125 [arXiv: arXiv:1210.0749].

[10] Zaslavskii O. B., *Phys. Rev.* D **84**, 024007 (2011) [arXiv:1104.4802].

[11] Bardeen J, Press W H and Teukolsky S A **178** (1972) 347 Astrophys. J.